B.Sc. (Maths) part II paper 4

Gradient, Divergence, Curl

① point function

A variable quantity which depends upon the co-ordinates of the point of a region of space is called a point function.

② vector point function

When each point of a plane & space a vector is arranged then the point function is called a vector point function

③ vector operator ∇. ~~The vector~~

The vector operator ∇ is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial x}$$

④ Gradient

The gradient of a scalar point function $\phi(x,y,z)$ is defined by

$$\text{grad}\,\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\& \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

= a vector quantity

(a) Divergence of a vector point function

Let $\vec{V}(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space then the divergence of $\vec{V}$ written as $\nabla\cdot\vec{V}$ or $\operatorname{div}\vec{V}$ is defined as

$$\operatorname{div}\vec{V} = \nabla\cdot\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}$$

$$= \vec{i}\cdot\frac{\partial \vec{V}}{\partial x} + \vec{j}\cdot\frac{\partial \vec{V}}{\partial y} + \vec{k}\cdot\frac{\partial \vec{V}}{\partial z}$$

= Scalar

(b) Curl of a vector point function

Def:- Let $\vec{V}(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space. Then the curl of $\vec{V}$ written as $\nabla\times\vec{V}$ or $\operatorname{curl}\vec{V}$ is defined as

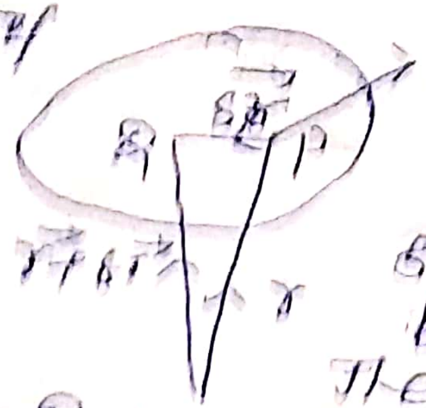
$$\operatorname{Curl}\vec{V} = \nabla\times\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$= \vec{i} \times \frac{\partial \vec{V}}{\partial x} + \vec{j} \times \frac{\partial \vec{V}}{\partial y} + \vec{k} \times \frac{\partial \vec{V}}{\partial z}$$

Geometrical interpretation of gradient

Theorem To prove that ∇f is a vector normal to the surface $f(x,y,z) = c$ where c is constant.

proof



Let the position vector of any point $P(x,y,z)$ on surface $f(x,y,z) = c$ be $\vec{r}$

Then $\vec{r} = x\vec{i} + y\hat{j} + z\hat{k}$

Let Q be a neighbouring point $(x+\delta x, y+\delta y)$ on the same level surface. Let the position vector of Q be $\vec{r} + \delta\vec{r}$

Then $\vec{r} + \delta\vec{r} = (x+\delta x)\vec{i} + (y+\delta y)\vec{j} + (z+\delta z)\vec{k}$

$\vec{PQ} = (\vec{r} + \delta\vec{r}) - \vec{r}$

$= (x+\delta x)\vec{i} + (y+\delta y)\vec{j} + (z+\delta z)\vec{k} - x\vec{i} - y\vec{j} - z\vec{k}$

$= \delta x\vec{i} + \delta y\vec{j} + \delta z\vec{k} = \delta r$ — (1)

Let $Q \to P$ then the line PQ tends to the tangent to the surface at P

Hence, in the limit (1) becomes

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ — (2)

It lies on the tangent plane to the surface at P

Now $\nabla f \cdot d\vec{r} = \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$

$\cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= df$$

But $f(x,y,z) = c$ ans

$$\therefore df = 0$$

$$\Rightarrow \nabla f \cdot d\vec{r} = 0$$

This shows that ∇f is perp.

$\therefore \nabla f$ is normal to the surface $f(x,y,z) = c$

Q:- To prove that

$$\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v})$$

$$\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}$$

Proof:- By definition we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\therefore \operatorname{div}(\vec{u}\times\vec{v}) = \nabla\cdot(\vec{u}\times\vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{u}\times\vec{v})$$

$$= \vec{i}\cdot\frac{\partial}{\partial x}(\vec{u}\times\vec{v}) + \vec{j}\cdot\frac{\partial}{\partial y}(\vec{u}\times\vec{v}) + \vec{k}\cdot\frac{\partial}{\partial z}(\vec{u}\times\vec{v})$$

$$= \vec{i}\cdot\left(\frac{\partial \vec{u}}{\partial x}\times\vec{v} + \vec{u}\times\frac{\partial \vec{v}}{\partial x}\right) +$$

$$+\vec{j}\cdot\left(\frac{\partial \vec{u}}{\partial y}\times\vec{v}+\vec{u}\times\frac{\partial \vec{v}}{\partial y}\right)$$

$$+\vec{k}\cdot\left(\frac{\partial \vec{u}}{\partial z}\times\vec{v}+\vec{u}\times\frac{\partial \vec{v}}{\partial z}\right)$$

$$=\left[\vec{i}\cdot\left(\frac{\partial\vec u}{\partial x}\times\vec v\right)+\vec j\cdot\left(\frac{\partial\vec u}{\partial y}\times\vec v\right)+\vec k\cdot\left(\frac{\partial\vec u}{\partial z}\times\vec v\right)\right]+\left[\vec i\cdot\left(\vec u\times\frac{\partial\vec v}{\partial x}\right)+\vec j\cdot\left(\vec u\times\frac{\partial\vec v}{\partial y}\right)+\vec k\cdot\left(\vec u\times\frac{\partial \vec v}{\partial z}\right)\right]$$

$$=\left[\left(\vec i\times\frac{\partial\vec u}{\partial x}\right)\cdot\vec v+\left(\vec j\times\frac{\partial\vec u}{\partial y}\right)\cdot\vec v+\left(\vec k\times\frac{\partial\vec u}{\partial z}\right)\cdot\vec v\right]-\left[\vec i\cdot\left(\frac{\partial\vec v}{\partial x}\times\vec u\right)+\vec j\cdot\left(\frac{\partial\vec v}{\partial y}\times\vec u\right)+\vec k\cdot\left(\frac{\partial\vec v}{\partial z}\times\vec u\right)\right]$$

as $\vec a\cdot(\vec b\times\vec c)=-\vec a\cdot(\vec c\times\vec b)$

$$=\left[\vec i\times\frac{\partial\vec u}{\partial x}+\vec j\times\frac{\partial\vec u}{\partial y}+\vec k\times\frac{\partial\vec u}{\partial z}\right]\cdot\vec v-\left[\left(\vec i\times\frac{\partial\vec v}{\partial x}\right)\cdot\vec u+\left(\vec j\times\frac{\partial \vec v}{\partial y}\right)\cdot\vec u+\left(\vec k\times\frac{\partial\vec v}{\partial z}\right)\cdot\vec v\right]$$

$$=\vec v\cdot\left\{\left(\vec i\frac{\partial}{\partial x}+\vec j\frac{\partial}{\partial y}+\vec k\frac{\partial}{\partial z}\right)\times\vec u\right\}-\vec u\cdot\left\{\left(\vec i\frac{\partial}{\partial x}+\vec j\frac{\partial}{\partial y}+\vec k\frac{\partial}{\partial z}\right)\times\vec v\right\}$$

$$=\vec v\cdot(\nabla\times\vec u)-\vec u\cdot(\nabla\times\vec v)$$

$$=\vec v\cdot\operatorname{curl}\vec u-\vec u\cdot\operatorname{curl}\vec v$$